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Feynman path-integral representation for scalar-wave propagation

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We propose a Feynman path-integral solution for wave propagation in an inhomogeneous medium.

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One of the long-standing unsolved problems in wave physics going back to Fresnel and Helmholtz is to find a general Feynman path integral for the scalar-wave equation in an inhomogeneous medium ([1], Chap. 20). In this Rapid Communication we propose a formal solution for the above-mentioned problem by writing a ν-dimensional space-time Feynman path-integral representation for the scalar-wave equation in a spatially variable inhomogeneous medium described by a refraction index $m(\mathbf{x})$ ($\mathbf{x}\in\mathbb{R}^{\nu-1}$).

Let us start our analysis by considering the corresponding Green function for an external point source

$$\frac{\partial^2}{\partial\tau^2}G((\mathbf{x},t);(\mathbf{y},t'))-m^{-2}(\mathbf{x})G((\mathbf{x},t);(\mathbf{y},t'))=\delta^{(\nu-1)}(\mathbf{x}-\mathbf{y})\delta(t-t') . \quad (1)$$

In order to write a space-time Feynman path-integral representation for the Green function Eq. (1) we follow Feynman by using the fifth-parameter technique by introducing a related Schrödinger wave equation with an initial point-source condition

$$i\frac{\partial}{\partial S}\psi((\mathbf{x},t);(\mathbf{y},t'),S)=\left[\frac{\partial^2}{\partial t^2}-m^{-2}(\mathbf{x})\Delta_{\mathbf{x}}\right]\psi((\mathbf{x},t);(\mathbf{y},t'),S) ,$$
$$\psi((\mathbf{x},t);(\mathbf{y},t'),0)=\delta^{(\nu-1)}(\mathbf{x}-\mathbf{y})\delta(t-t') , \quad (2)$$
$$\psi((\mathbf{x},t);(\mathbf{y},t'),\infty)=0 .$$

At this point we remark the following identity between the Schrödinger wave equation (2) and the scalar-wave Green function Eq. (1):

$$G((\mathbf{x},t);(\mathbf{y},t'))=-i\int_0^\infty dS\,\psi((\mathbf{x},t);(\mathbf{y},t'),S) . \quad (3)$$

In order to write a path integral for the associated Schrödinger equation (2) we consider the solution in the operator-matrix form (the Feynman-Dirac propagator) [1].

$$\psi((\mathbf{x},t);(\mathbf{y},t'),S)=\langle(\mathbf{x},t)|e^{iS\mathcal{L}}|(\mathbf{y},t')\rangle , \quad (4)$$

where $\mathcal{L}$ denotes the D'Alembert wave operator for $m(\mathbf{x})$. As in quantum mechanics we write the propagator Eq. (4) as an infinite product of short-time S propagations

$$\langle(\mathbf{x},t)|e^{iS\mathcal{L}}|(\mathbf{y},t')\rangle=\lim_{N\to\infty}\prod_{i=j}^{N}\int d^{\nu-1}\mathbf{x}_i dt_i\langle(\mathbf{x}_i,t_i)|e^{i(S/N)\mathcal{L}}|(\mathbf{x}_{i-j},t_{i-j})\rangle . \quad (5)$$

The standard short-time expansion in the S parameter for the D'Alembert wave operator is given by ([2], Chap. 10)

$$\lim_{S\to0+}\langle(\mathbf{x}_i,t_i)|e^{iS\mathcal{L}}|(\mathbf{x}_{i-j},t_{i-j})\rangle=\lim_{S\to0+}\int(d^{\nu-1}\rho_i)(dw_i)\exp\{iS[-w_i^2+m^{-2}(\mathbf{x}_i)\rho_i^2]\}\exp[i\rho_i(\mathbf{x}_i-\mathbf{x}_{i-t})+iw_i(t_i-t_{i-t})] . \quad (6)$$

If we substitute Eq. (6) into Eq. (5) and take the Feynman limit of $N\to\infty$, we will obtain the following weighted path-integral representation after evaluating the (ρ_i,w_i) Gaussian integrals of the representation Eq. (6) for the right-hand side of Eq. (5):

$$\langle(\mathbf{x},t)|e^{iS\mathcal{L}}|(\mathbf{y},t')\rangle=\int\left[\prod_{\substack{0\le\sigma\le S\\ t(0)=t;t(S)=t'}}dt(\sigma)\right]\left[\prod_{\substack{0\le\sigma\le S\\ \mathbf{r}(0)=\mathbf{x};\mathbf{r}(S)=\mathbf{y}}}d\mathbf{r}(\sigma)(m(\mathbf{r}(\sigma)))^{\nu-1}\right]\times\exp\left\{i\int_0^S\left[\frac{dt(\sigma)}{d\sigma}\right]^2d\sigma-i\int_0^S m^2(\mathbf{r}(\sigma))\left|\frac{d\mathbf{r}(\sigma)}{d\sigma}\right|^2d\sigma\right\} , \quad (7)$$

where $t(\sigma)$ and $\mathbf{r}(\sigma)$ are the Feynman-Brownian space-time ray trajectories connecting the initial and final space-time points $(\mathbf{x},t)$ and $(\mathbf{y},t')$.

It is instructive to remark that the $t(\sigma)$ Feynman path integral is exactly soluble [1]. As a consequence we finally obtain our proposed space-time path-integral representation for Eq. (1)

$$G((\mathbf{x},t),(\mathbf{y},t'))=\int_0^\infty dS\, e^{i(t-t')^2/S}\int \left[\prod_{\substack{0\le\sigma\le S\\ \mathbf{r}(0)=\mathbf{x},\mathbf{r}(S)=\mathbf{y}}} d\mathbf{r}(\sigma)(m(\mathbf{r}(\sigma)))^{\nu-1}\right]\exp\left\{-i\int_0^S m^2(\mathbf{r}(\sigma))\left|\frac{d\mathbf{r}(\sigma)}{d\sigma}\right|^2 d\sigma\right\}. \quad (8)$$

For the simplest case of a constant refraction index $m^2(\mathbf{x})=1/C_0^2$ the Feynman path integral Eq. (8) is exactly solved and yields as a result the usual Lienard-Weichert potential after introducing the retarded causality condition $(\mathbf{x}-\mathbf{y})^2>c_0^2(t-t')\Longrightarrow G((\mathbf{x},t),(\mathbf{y},t'))\equiv 0$.

We point out the usefulness of Eq. (8) to obtain explicit formulas for wave propagation in a random medium [2,3], since the $\{m^2(\mathbf{x})\}$ random variable appears explicitly in the proposed formulas, Eq. (8). For instance, the averaged Green function Eq. (1) for a random medium with Gaussian statistics ([1], Chap. 28)

$$\langle m^2(\mathbf{x}_1)m^2(\mathbf{x}_2)\rangle=K(|\mathbf{x}_1-\mathbf{x}_2|) \quad (9)$$

will lead us to consider the following polaronlike Feynman path integral as an effective expression for the above-cited averaged Green function ([1], Chap. 21):

$$\langle G((\mathbf{x},t);(\mathbf{y},t'))\rangle\cong\int_0^\infty dS\, e^{i(t-t')^2/S}\int\left[\prod_{\substack{0\le\sigma\le S\\ \mathbf{r}(0)=\mathbf{x},\mathbf{r}(S)=\mathbf{y}}} d\mathbf{r}(\sigma)\right]\exp\left\{-\int_0^S d\sigma\int_0^S d\sigma'\left|\frac{d\mathbf{r}(\sigma)}{d\sigma}\right|^2 K(|\mathbf{r}(\sigma)-\mathbf{r}(\sigma')|)\times\left|\frac{d\mathbf{r}(\sigma')}{d\sigma'}\right|^2\right\}, \quad (10)$$

which was obtained after using the approximation for the Feynman-Brownian ray path measure

$$\prod_{\substack{0\le\sigma\le S\\ \mathbf{r}(0)=\mathbf{x},\mathbf{r}(S)=\mathbf{y}}}[d\mathbf{r}(\sigma)]m^{\nu-1}(\mathbf{r}(\sigma))\cong\prod_{\substack{0\le\sigma\le S\\ \mathbf{r}(0)=\mathbf{x},\mathbf{r}(S)=\mathbf{y}}}[d\mathbf{r}(\sigma)]\,. \quad (11)$$

Let us point out that the approximation Eq. (11) is exact for $|\mathbf{x}-\mathbf{y}|$ much larger than the length scale of the medium randomness [2,3].

Work on scalar-wave propagation in a spatially turbulent medium [3] in this Feynman path-integral approach will be reported elsewhere.

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